Lesson 22. Double Integrals over Rectangles

0 Warm up

Example 1. Find the value of

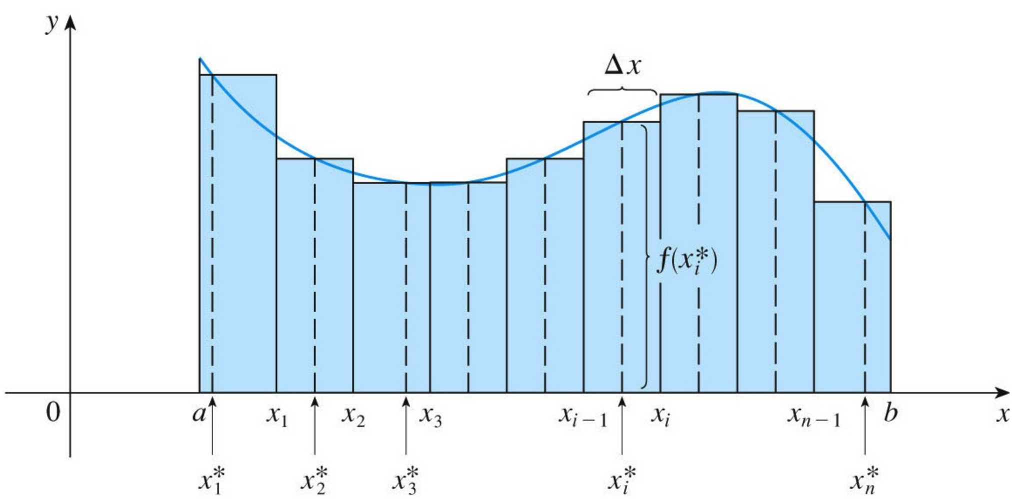
a. $\sum_{i=2}^{4} i =$

b. $\sum_{i=1}^{3} \sum_{j=1}^{2} ij =$

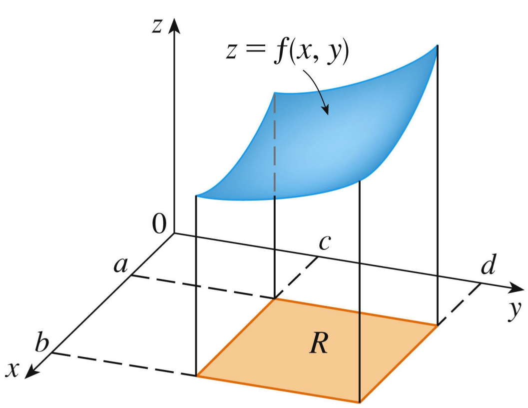
1 Review: area and integrals

- The definite integral of a single-variable function:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$



2 Volume and double integrals



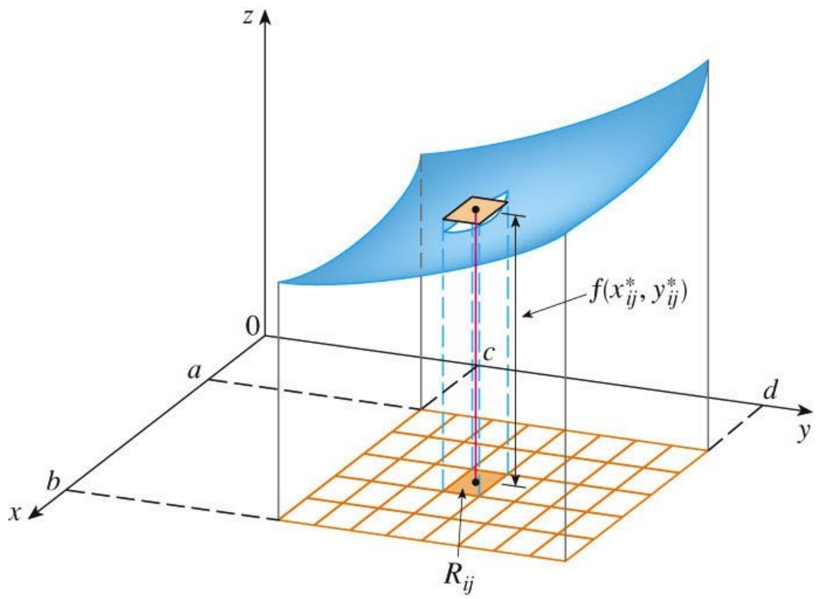
- Let R be a rectangle in the xy-plane:

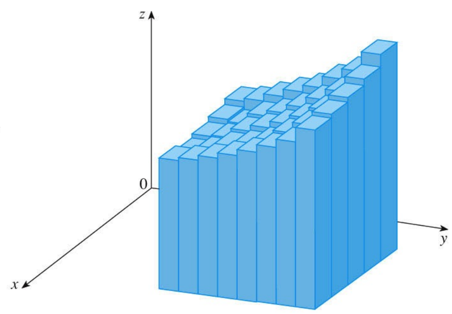
$$R = [a,b] \times [c,d] = \left\{(x,y) : a \le x \le b, c \le y \le d\right\}$$

- Let $f(x,y)$ be a function of two variables
- What is the volume of the solid above R and below the graph of f?

- Idea:

- Divide R into subrectangles of equal area ΔA
  - Grid with m columns (x-direction) and n rows (y-direction)
- For each subrectangle R_{ij}:
  - Choose a **sample point** (x_{ij}^*, y_{ij}^*)
  - Compute the volume of the (thin) rectangular box with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$.
- Add the volumes of all these rectangular boxes



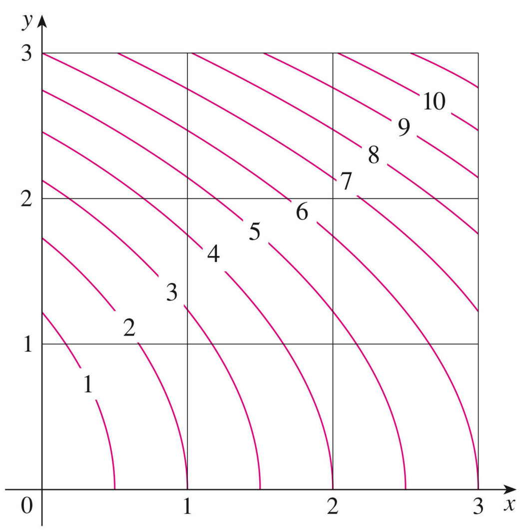


- Estimated volume:
  - This is called a **double Riemann sum**
- The **double integral** of f over the rectangle R is

- How do we choose sample points in each subrectangle?
  - Upper right corner
  - Lower left corner
  - **Midpoint rule**: center of subrectangle
- If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

Example 2. Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below $f(x, y) = 16 - x^2 - 2y^2$. Use a Riemann sum with $m = 2$ and $n = 2$. Use the upper right corners as sample points.

Example 3. Below is a contour map for a function f on the square $R = [0, 3] \times [0, 3]$. Use a Riemann sum with $m = 3$ and $n = 3$ to estimate the value of $\iint_R f(x, y)\, dA$. Use the midpoint rule to take sample points.



3 Average value

- The **average value** of a function of two variables defined on a rectangle R is

Example 4. Estimate the average value of the function f in Example 3 on R.

4 Iterated integrals

- **Partial integration** with respect to x: $\int_a^b f(x, y)\, dx$
  - Regard y as a constant (i.e., fixed, coefficient, etc.)
  - Integrate $f(x, y)$ with respect to x from $x = a$ to $x = b$
  - Results in an expression in terms of y
- Partial integration with respect to y defined in a similar way
- **Iterated integrals**: work from the inside out
  - $\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$
    - Integrate first with respect to x from $x = a$ to $x = b$ (keeping y constant)
    - Integrate resulting expression in y with respect to y from $y = c$ to $y = d$
  - $\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$
    - Integrate first with respect to ☐ from ☐
    - Integrate resulting expression in ☐ with respect to ☐ from ☐

Example 5. Evaluate $\int_0^3 \int_1^2 x^2 y\, dy\, dx$.

Example 6. Evaluate $\int_1^2 \int_0^3 x^2 y\, dx\, dy$.

- **Fubini's theorem for rectangles.** If $R = [a, b] \times [c, d]$, then:

  - (f needs to satisfy some conditions, e.g. f is continuous on R)
  - Double integrals over rectangles can be evaluated using iterated integrals
  - Order of integration does not matter!

Example 7. Evaluate $\iint_R (x - 3y^2)\, dA$, where $R = [0, 2] \times [1, 2]$.

Example 8. Find the volume of the solid that is bounded by the surface $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Example 9. Evaluate $\int_0^1 \int_0^1 ye^{xy}\,dy\,dx$.